Mathematics Extended Essay
2012-2014

Extended Essay

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I. Introduction

Since ancient times, rainbows have been the subject of mankind's fascination. From the times of Aristotle to the times of Sir Isaac Newton, many have tried to find the scientific explanation to the visual phenomenon known as the rainbow. And while the study of rainbows in itself falls into the topic of optical physics, the calculations behind the study of rainbows relies mostly in the application of calculus.

While I do have a personal interest in rainbows and optical phenomena, the theoretical calculations behind them have begun to fascinate me. The differential calculus and geometry aspect of the calculations seem mostly simple, as I have had learned differential calculus since tenth grade, the complexity comes in understanding the concept of how rainbows are formed. I think that the more mathematical aspect of rainbows, or optical physics in general, is a great area to research in.

In this essay, I shall study the use of calculus to calculate the angles needed to create a variety of rainbows, from primary rainbows to higher order rainbows. And to see if there is any real pattern between the formation of rainbows and the order of the colors that occur in that rainbow as well as its angles.

II. An Introduction to Rainbows

Fig. 1: A primary rainbow[1]

Before the topic of rainbows can be assessed, the theory behind how a rainbow is made must first be learned. Rainbows are formed when light is refracted and reflected inside a water droplet. Reflection of light is when light is reflected off a surface and into a new path, while refraction of light is the bending of light when it enters a medium different from its previous one. From this phenomena, the Snell's law can be used as a method of calculation.

Snell's Law is the rule in which we could find the refractive index of an object through finding its incident angle (angle at which a ray of light enters or exits a medium) and refracted angle (angle at which a ray of light is refracted as it enters or exits a medium). Below is an example of a refracted ray of light when it goes from one medium to another:

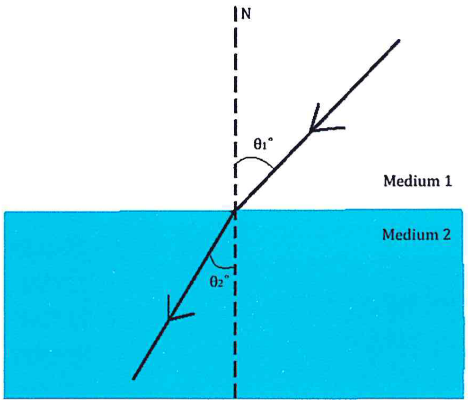

Fig. 2: Light going through two different mediums

Where N is the normal to Medium 2, θ_1 and θ_2 are the angles of the ray of light in respect to the normal and n_1 and n_2 are the refractive indices of Medium 1 and Medium 2 respectively.

[1] http://www.atoptics.co.uk/rainbows/images1/kk31_r1_c1.jpg

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According to WolframResearch[2], the Snell's law states that:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

or,

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{n_2}{n_1}$$

I shall use this law when calculating for the value of the angle of deviation.

As refraction and reflection occur inside the water droplet, the angle of deviation of the ray of light can be calculated. The angle of deviation is defined as how much a ray of light is deviated (reflected) from its original path. In other words, the angle of deviation, $D(\alpha)$, is the amount of clockwise, or counter-clockwise, rotation that the light undergoes in its path into and out of the raindrop. To further explain $D(\alpha)$, we turn to the diagram below:

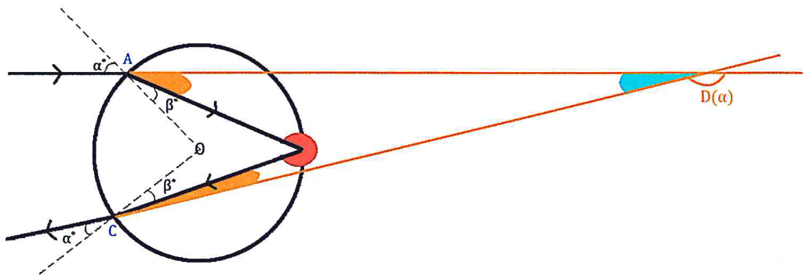

Fig. 3: The formation of rainbow, including $D(\alpha)$

When a line is extended from the entry ray and the exiting ray, the angle that forms when the two lines intersect is the $D(\alpha)$. This value of $D(\alpha)$ can be calculated by finding the value of the angle marked by blue. To find the blue section, we must first find the sections marked with orange and the angle marked red.

As there are many paths that can be undertaken by the sunlight into the water droplet (as there are multiple rays of light and distribution of light is random), we have to find the minimum of angle of deviation. According to Atmospheric Optics[3], the bow is at its brightest at the minimum angle of deviation because more rays of light are concentrated in that angle. The higher the order of rainbows (named for the amount of internal reflection the light experiences. For example, first-order rainbows experience one reflection, second-order rainbows experience two, and so on), however, the fainter they become.

2 http://scienceworld.wolfram.com/physics/SnellsLaw.html
3 (Cowley)

III. Primary Rainbows

Primary rainbows occur when sunlight is refracted, then reflected once inside a raindrop. The path of the ray of light is illustrated using this picture:

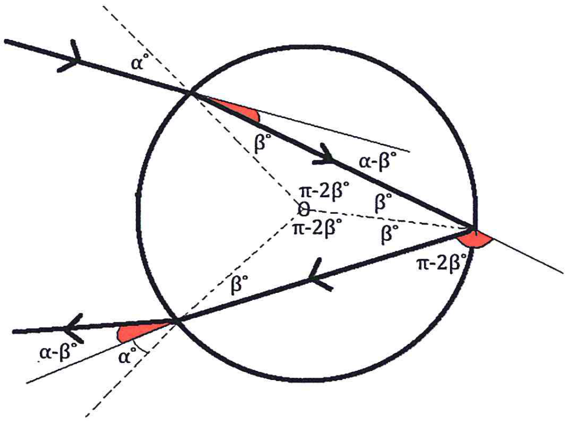

Fig. 2: The refraction and reflection of light in a raindrop.

Where alpha (α) is the incident angle, beta (β) is the refracted angle.

Therefore, to obtain $D(\alpha)$, we can calculate it as:

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta)$$

$$D(\alpha) = \alpha - \beta + \pi - 2\beta + \alpha - \beta$$

$$D(\alpha) = \pi + \alpha + \alpha - \beta - \beta - 2\beta$$

$$D(\alpha) = \pi + 2\alpha - 4\beta$$

We shall use this formula to obtain the angle of elevation. The angle of elevation is simply the angle at which an observer can view a rainbow. To allow ease, this situation can be represented by a simple diagram:

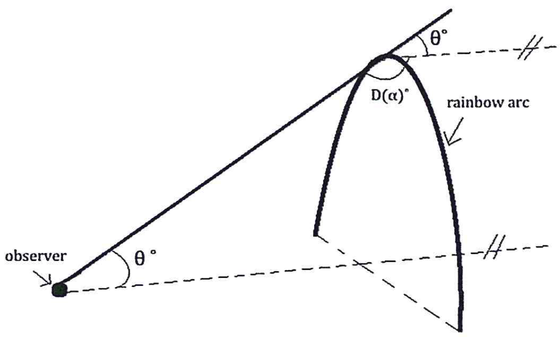

Fig. 3: A rainbow and an observer

To find the minimum angle of elevation, we must subtract $D(\alpha)$ from π, because, as it can be seen in the diagram, the angle of elevation, represented by $\theta°$, and $D(\alpha)°$ make up a straight line.

Before that, we must find $D(\alpha)$. Because we are looking for the minimum angle of deviation, we may use differentiation to find the minimum of $D(\alpha)$, where $D'(\alpha) = 0$.

First, we shall use Snell's Law to find β in terms of α. For this calculation, I shall be using the second variation mentioned above of Snell's Law.

$$\sin(\alpha) = n\sin(\beta)$$

Where n equals to the refractive index of the object, α is the angle of incidence and β is the angle of refraction. Using this equation we can find β in terms of α.

$$\sin(\alpha) = n\sin(\beta)$$

$$\frac{\sin(\alpha)}{n} = \sin(\beta)$$

$$\arcsin(\sin(\beta)) = \arcsin\left(\frac{sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right) \quad (1)$$

Then we substitute equation (1) into the variable β in the formula calculating for $D(\alpha)$.

$$D(\alpha) = \pi + 2\alpha - 4\arcsin\left(\frac{\sin(\alpha)}{n}\right) \quad (2)$$

We then differentiate this equation, and since we are trying to find the minimum angle of deviation, then $D'(\alpha) = 0$. First, we must find the derivative of $D(\alpha)$ in respect to α.

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{d\alpha}\left(\arcsin\left(\frac{\sin(\alpha)}{n}\right)\right) = \frac{\cos(\alpha)}{n} \times \frac{1}{\sqrt{1-\frac{sin^2(\alpha)}{n^2}}}$$

$$\frac{d}{d\alpha}\left(\arcsin\left(\frac{\sin(\alpha)}{n}\right)\right) = \frac{\cos(\alpha)}{n\sqrt{1-\frac{sin^2(\alpha)}{n^2}}}$$

$$D'(\alpha) = \frac{d}{d\alpha}(\pi) + \frac{d}{d\alpha}(2\alpha) - \frac{d}{d\alpha}\left(\arcsin\left(\frac{\sin(\alpha)}{n}\right)\right)$$

$$D'(\alpha) = 2 - 4\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}$$

$$D'(\alpha) = 0$$

$$0 = 2 - 4\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}$$

After that, we move $4\frac{\cos(\alpha)}{n\sqrt{1-\frac{sin^2(\alpha)}{n^2}}}$ to the left hand side, so that:

$$4\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}} = 2$$

We then try to simplify this equation so that we can find α in terms of n.

$$\cos(\alpha) = \frac{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}{2}$$

$$\cos^2(\alpha) = \frac{n^2\left(1 - \frac{sin^2(\alpha)}{n^2}\right)}{4}$$

$$4\cos^2(\alpha) = n^2\left(1 - \frac{sin^2(\alpha)}{n^2}\right)$$

$$4\cos^2(\alpha) = n^2 - sin^2(\alpha)$$

$$n^2 = 4\cos^2(\alpha) + sin^2(\alpha)$$

$$n^2 = 3\cos^2(\alpha) + 1$$

$$3\cos^2(\alpha) = n^2 - 1$$

$$\cos^2(\alpha) = \frac{n^2 - 1}{3}$$

$$\cos(\alpha) = \sqrt{\frac{n^2 - 1}{3}}$$

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{3}}\right) \qquad (3)$$

According to TutorVista[4] the refractive index of light in water is around 1.333.

$$\alpha = \arccos\left(\sqrt{\frac{1.333^2 - 1}{3}}\right)$$

$$\alpha \approx 1.03691 \text{ rads}$$

However, if the angle is calculated in terms of radians, it is harder to envision the size of the angle. Thus, for ease of calculating and comparing results, we shall change the value in radians to degrees.

$$\alpha = \frac{1.03691}{2\pi} \times 360^\circ$$

$$\alpha \approx 59.41047^\circ$$

From here onwards, all the angles shall be calculated in terms of degrees rather than radians.

We then incorporate this value for α into equation (2)

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(59.58492^\circ)}{1.33}\right)$$

$$\beta \approx 40.42155^\circ$$

We can input both of these values into the formula for the angle of deviation, which we had proved earlier. Thus, using these values, we can find the minimum angle of deviation.

$$D(\alpha) = \pi + 2(59.58492^\circ) - 4(40.42155^\circ)$$

$$D(\alpha) = 137.48364^\circ$$

To find the angle of elevation, we simply subtract the value we obtained from π. Using the value obtained, we will be able to find θ (the angle of elevation). This is simply done by subtracting the value of $D(\alpha)$ from π.

$$\theta = \pi - 137.48364 \approx 42.5^\circ$$

The answer is rounded off to 42.5˚.

Note that this result is only used as a sort of benchmark on the angle of elevation of the rainbow as a whole. This angle of elevation is from the observer's viewpoint to

4 (NCS Pearson, 2013)

the highest point of the arc of the rainbow. This value is set to be the maximum value (because it is taken from the highest point of the rainbow) that the value of θ can reach.

We can also use these formulas to find α, β, $D(\alpha)$ as well as the θ when the viewer sees a particular color. Using this method, we will also find out why the rainbow is arranged as: red, orange, yellow, green, blue, indigo, and violet.

Colors in a Primary Rainbow

The refractive indices for the seven colors of the visible spectrum are given as shown below. The data is obtained by comparing the values from Numericana[5], which does not have a very complete data set, with the data set from Philip Laven[6], which covers quite a broad range of the rainbow color spectra. One way to do this is by comparing the wavelengths of the colors needed from both sets of data, and the data stated in the table below is the most optimal result that I can achieve from both sets of data by comparing the wavelengths of the light spectrum.

Colors	Refractive Index
Red	1.33257
Orange	1.33322
Yellow	1.33472
Green	1.33659
Blue	1.33903
Indigo	1.34235
Violet	1.34451

Fig. 4: Table of refractive indices

First, we shall try to find the angle of elevation of the first color, which is red. To do this, we use equation (3), used in the previous calculation to find for α when the refractive index is 1.333.

[5] (Michon, 2013)
[6] (Laven, 2012)

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{3}}\right)$$

And then we substitute in the value of the refractive index for red light in water (because the medium is a water droplet):

$$\alpha = \arccos\left(\sqrt{\frac{1.33257^2 - 1}{3}}\right)$$

$$\alpha \approx 59.43547°$$

Then, using equation (1), we use this value of α to find the incident angle, β.

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(59.43547°)}{1.33257}\right)$$

$$\beta \approx 40.25290°$$

And finally, we substitute α and β into (2).

$$D(\alpha) = \pi + 2(59.43547°) - 4(40.25290°)$$

$$D(\alpha) = 137.85934°$$

$$\theta = \pi - 137.85934° \approx 42.1°$$

By using approximation, we can find that θ, the angle of elevation, at which the red light is viewed is approximately at 42.1° from the ground. We then use the same methods to get the angle of elevation for other colors.

For example, to find the angle of elevation of orange, we simply repeat the same steps taken previously, and only changing the refractive index to 1.33322, the value stated in the table above.

$$\alpha = \arccos\left(\sqrt{\frac{1.33322^2 - 1}{3}}\right)$$

$$\alpha \approx 59.39769°$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(59.39769°)}{1.33322}\right)$$

$$\beta \approx 40.21038°$$

$$D(\alpha) = \pi + 2(59.39769°) - 4(40.21038°)$$

$$D(\alpha) = 137.95386°$$

$$\theta = \pi - 137.95386° \approx 42.0°$$

The same can also be done for yellow light, for the refractive index of 1.33472.

$$\alpha = \arccos\left(\sqrt{\frac{1.33472^2 - 1}{3}}\right)$$

$$\alpha \approx 59.31055°$$

$$\beta = \arcsin\left(\frac{\sin(59.31055°)}{1.33472}\right)$$

$$\beta \approx 40.11243°$$

$$D(\alpha) = \pi + 2(59.31055°) - 4(40.11243°)$$

$$D(\alpha) \approx 138.17138°$$

$$\theta = \pi - 137.17138° \approx 41.8°$$

We then repeat this process for the rest of the colors. The results for the calculations of each of the colors are shown in the table below:

Colors	Angle of elevation/°
Red	42.1
Orange	42.0
Yellow	41.8
Green	41.6
Blue	41.5
Indigo	40.7
Violet	40.4

Fig. 5: Table for angles of elevation (θ)

Looking at the values in the table above, it can be seen that the value of the angle of elevation decreases. This signifies that each color gets closer to the ground, going by the order of red as the furthest from the ground and violet as the closest.

Therefore, these calculations prove that the order of colors of the rainbow are red, orange, yellow, green, blue, indigo and violet according to the decreasing angles. There is also a slight pattern between the angles, with red and orange, and green and blue placed close together, while there is a big gap between blue and indigo. I shall see if this result shall pertain to the next order of rainbows.

IV. Secondary Rainbows

Secondary rainbows are formed when the ray of light experiences two total internal reflections inside the droplet of water instead of one. This occurrence is illustrated in this diagram:

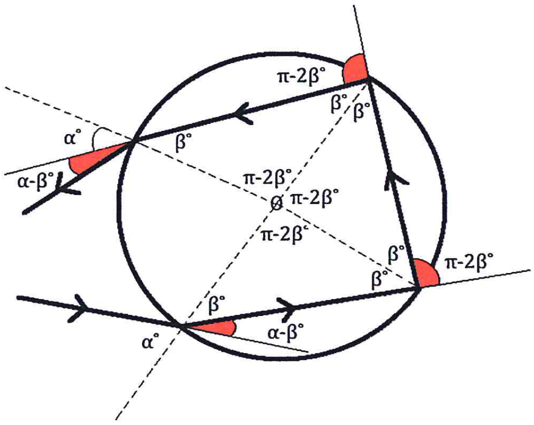

Fig. 5: The formation of secondary rainbows

To calculate the angle of deviation for a secondary rainbow, we calculate for the total counter-clockwise rotation of the light ray as it goes along its path, which is marked as red. Therefore the total angle of deviation can be calculated as:

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\pi - 2\beta) + (\alpha - \beta)$$

$$D(\alpha) = \alpha - \beta + \pi - 2\beta + \pi - 2\beta + \alpha - \beta$$

$$D(\alpha) = \pi + \pi + \alpha + \alpha - \beta - \beta - 2\beta - 2\beta$$

$$D(\alpha) = 2\pi + 2\alpha - 6\beta$$

We then use equation (1) once again to substitute for the variable β in the formula to find the angle of deviation.

$$D(\alpha) = 2\pi + 2\alpha - 6\arcsin\left(\frac{\sin(\alpha)}{n}\right) \qquad (4)$$

Since we are trying to find the minimum angle of deviation, then the derivative of $D(\alpha)$ will be equal to 0.

$$D'(\alpha) = 2 - 6\frac{\cos(\alpha)}{n\sqrt{1 - \frac{\sin^2(\alpha)}{n^2}}} = 0$$

We then simplify this equation so that we can find a formula for α in terms of n, which is the refractive index. As what we have done previously, we will use the value of α to find the value of β and the angles of deviation and elevation.

$$2 - 6\frac{\cos(\alpha)}{n\sqrt{1 - \frac{\sin^2(\alpha)}{n^2}}} = 0$$

$$6\frac{\cos(\alpha)}{n\sqrt{1 - \frac{\sin^2(\alpha)}{n^2}}} = 2$$

$$3\cos(\alpha) = n\sqrt{1 - \frac{\sin^2(\alpha)}{n^2}}$$

$$9\cos^2(\alpha) = n^2 - \sin^2(\alpha)$$

$$n^2 = 9\cos^2(\alpha) + \sin^2(\alpha)$$

$$n^2 = 8\cos^2(\alpha) + 1$$

$$8\cos^2(\alpha) = n^2 - 1$$

$$\cos^2(\alpha) = \frac{n^2 - 1}{8}$$

$$\cos(\alpha) = \pm\sqrt{\frac{n^2 - 1}{8}}$$

The same as with primary rainbows, we will only consider the positive value of the square root.

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{8}}\right) \qquad (5)$$

We shall use this formula to find the minimum angle of deviation. As with the previous calculation, we use the refractive index of water to find the value of α.

$$\alpha = \arccos\left(\sqrt{\frac{1.333^2 - 1}{8}}\right)$$

$$\alpha \approx 71.93955°$$

Then, substituting the value of α equation (1) once again:

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(71.84272°)}{1.333}\right)$$

$$\beta \approx 45.46578°$$

And finally, substitute the values of both α and β into equation (4).

$$D(\alpha) = 2\pi + 2\alpha - 6\beta$$

$$D(\alpha) = 2\pi + 2(71.84272°) - 6(45.46578°)$$

$$D(\alpha) = 230.89076°$$

To find for θ, we shall subtract the value of $D(\alpha)$ from π, as done previously for the primary rainbow.

$$\theta = \pi - 230.10126° \approx -50.1°$$

Therefore, the angle of elevation at which a secondary rainbow can be seen is 50.1˚. For this calculation, the negative is not taken into account, as the angle of elevation will always be positive for the rainbow to be visible.

Notice that the angle of elevation is higher than that of the primary rainbow. This value proves that the secondary rainbow is placed higher in the sky than the primary rainbow, which many photographic evidences have proven. One of those photographic evidences is this picture:

Fig. 6: A primary and a secondary rainbow[7]

[7] http://www.wired.com/geekdad/wp-content/uploads/2012/09/IMG_0267-660x492.jpg

According to this picture, the secondary rainbow is placed higher than the primary rainbow and has an inverted order of colors. Thus, instead of going from red to violet, the secondary rainbow goes from violet to red.

We then use the same methods to calculate for the angle of elevation of the different colors inside the spectrum. Using this method, we shall try to prove that the order of colors in a secondary rainbow is inverted from that of the primary rainbow.

Colors in a Secondary Rainbow

Using the values of the refractive index of the various colors mentioned previously, we could find the angle of elevation for each color in the visible spectrum according to the formation of the secondary rainbow.

For red, we shall repeat the process done on the primary rainbow using the equations (4) and (5) instead of (2) and (3). Therefore, using equation (5), we will find for the value of α.

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{8}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33257^2 - 1}{8}}\right)$$

$$\alpha \approx 71.85658^\circ$$

And then substitute the value into equation (1).

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(71.85658^\circ)}{1.33257}\right)$$

$$\beta \approx 45.48920^\circ$$

And finally substitute both values into equation (4).

$$D(\alpha) = 2\pi + 2\alpha - 6\beta$$

$$D(\alpha) = 2\pi + 2(71.85658^\circ) - 6(45.48920^\circ)$$

$$D(\alpha) = 230.77796^\circ$$

$$\theta = \pi - 230.77796° \approx -50.8°$$

And because the negative is not taken into account, the angle of elevation is simply 50.8°.

We shall then use the same method to find for the θ of orange:

$$\alpha = \arccos\left(\sqrt{\frac{1.33322^2 - 1}{8}}\right)$$

$$\alpha \approx 71.83562°$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(71.83562°)}{1.33322}\right)$$

$$\beta \approx 45.45381°$$

$$D(\alpha) = 2\pi + 2(71.83562°) - 6(45.45381°)$$

$$D(\alpha) = 230.94838°$$

$$\theta = \pi - 230.94838° \approx -50.9°$$

And yellow:

$$\alpha = \arccos\left(\sqrt{\frac{1.33472^2 - 1}{8}}\right)$$

$$\alpha \approx 71.78730°$$

$$\beta = \arcsin\left(\frac{\sin(71.78730°)}{1.33472}\right)$$

$$\beta \approx 45.37234°$$

$$D(\alpha) = 2\pi + 2(71.78730°) - 6(45.37234°)$$

$$D(\alpha) = 231.34056°$$

$$\theta = \pi - 231.34056° \approx -51.3°$$

And then we find for the rest of the colors, the results are then put into the table below:

Colors	Angle of elevation/°	
	Primary rainbows	Secondary rainbows
Red	42.1	50.8
Orange	42.0	50.9
Yellow	41.8	51.3
Green	41.6	51.8
Blue	41.5	52.5
Indigo	40.7	53.3
Violet	40.4	53.9

From these results, we can see that the values of θ increase as it goes from red to violet. This calculation gives proof that the secondary rainbow does, in fact have an inverted order of colors in contrast to the primary rainbow. However, there is a greater gap between green and blue, in contrast to the primary rainbow, and the angles are all generally further apart. The arc may seem higher.

V. Tertiary Rainbows

I have gone over how to calculate for the values of the minimum angles of deviation and the angles of elevation for the primary and secondary rainbow, but are there other forms of rainbows? Studies have shown that there can be, in fact, higher order of rainbows. One such rainbow is the tertiary rainbow. Tertiary rainbows occur when the light ray undergoes a total of three reflections inside the water droplet. This occurrence is demonstrated by the diagram below:

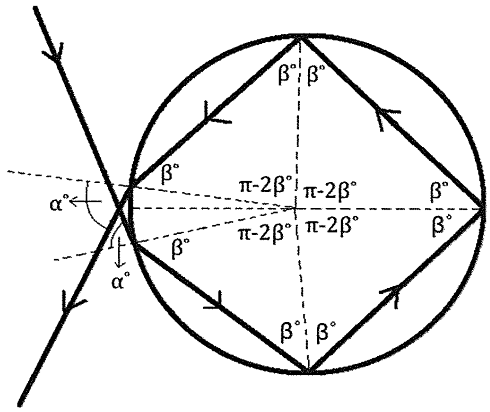

Fig. 7: The formation of tertiary rainbows

This diagram (in reference to the diagram from Harvard Physics[8]) shows the three internal reflections that occur inside the water droplet. The same method applies when calculating for the $D(\alpha)$ as with our previous calculations. We simply find the sum of all counter-clockwise rotation that the ray of light experiences.

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\pi - 2\beta) + (\pi - 2\beta) + (\alpha - \beta)$$

$$D(\alpha) = \alpha - \beta + \pi - 2\beta + \pi - 2\beta + \pi - 2\beta + \alpha - \beta$$

$$D(\alpha) = \alpha + \alpha - \beta - \beta - 2\beta - 2\beta - 2\beta + \pi + \pi + \pi$$

$$D(\alpha) = 2\alpha - 8\beta + 3\pi \qquad (6)$$

And then to find the minimum angle of deviation, we differentiate $D(\alpha)$ and find for α when $D'(\alpha) = 0$.

$$D(\alpha) = 2\alpha - 8\arcsin\left(\frac{\sin(\alpha)}{n}\right) + 3\pi$$

[8] https://www.physics.harvard.edu/uploads/files/undergrad/probweek/sol81.pdf

$$D'(\alpha) = 2 - 8\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}$$

$$2 - 8\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}} = 0$$

$$8\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}} = 2$$

$$\cos(\alpha) = \frac{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}{4}$$

$$\cos^2(\alpha) = \frac{n^2\left(1 - \frac{sin^2(\alpha)}{n^2}\right)}{16}$$

$$16\cos^2(\alpha) = n^2 - sin^2(\alpha)$$

$$n^2 = 16\cos^2(\alpha) + sin^2(\alpha)$$

$$n^2 = 15\cos^2(\alpha) + 1$$

$$15\cos^2(\alpha) = n^2 - 1$$

$$\cos^2(\alpha) = \frac{n^2 - 1}{15}$$

$$\cos(\alpha) = \pm\sqrt{\frac{n^2 - 1}{15}}$$

Again, we take the positive value of $\cos(\alpha)$.

$$\cos(\alpha) = \sqrt{\frac{n^2 - 1}{15}}$$

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{15}}\right) \qquad (7)$$

Finally, just as with the primary and secondary rainbows, we find for the angle of elevation, θ, when the refractive index is 1.333.

$$\alpha = \arccos\left(\sqrt{\frac{1.333^2 - 1}{15}}\right)$$

$$\alpha = 76.84537^{\circ}$$

Substitute the value of α into equation (1):

$$\beta = \arcsin\left(\frac{\sin(76.84537^{\circ})}{1.333}\right)$$

$$\beta \approx 46.92851^{\circ}$$

And then we put in both values into equation (6).

$$D(\alpha) = 2(76.84537^{\circ}) - 8(46.92851^{\circ}) + 3\pi$$

$$D(\alpha) = 318.26266^{\circ}$$

Finally, we subtract the value of $D(\alpha)$ from π.

$$\theta = \pi - 318.26266^{\circ} \approx -138.3^{\circ}$$

Since the value of θ is more than 90˚ (we ignore the negative sign as with the one for secondary rainbow), the rainbow is then in the opposite direction of the primary rainbow. As we have only calculated for when the observer faces a certain direction. We then find the angle of elevation relative to the direction of the rainbow.

$$\theta_e = \pi - 138.3 = 41.7^{\circ}$$

Where θ_e is the angle of elevation of the rainbow

Therefore, the angle of elevation is 41.7˚.

Though tertiary rainbows are possible theoretically, it is very difficult to find them in practice. Tertiary rainbows are faint in terms of brightness. Thus there is little to no photographical evidence, in contrast to the primary and secondary rainbows, of tertiary rainbows ever occurring in real life.

We shall use the same steps as the calculations done previously to find the angles of elevation for all seven colors. And thus, we shall also find the order of the colors of the tertiary rainbow.

Colors in a Tertiary Rainbow

Again, we use the values of the refractive index mentioned previously to find the values. Also, we shall be using equations (6) and (7) instead of the previous equations. We shall then use equation (7) to find for α.

For red, we shall simply substitute in the value of its refractive index into n.

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{15}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33257^2 - 1}{15}}\right)$$

$$\alpha \approx 76.85525^\circ$$

And then, as with previous calculations, we substitute the value of α into equation (1).

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(76.85525^\circ)}{1.33257}\right)$$

$$\beta \approx 46.95076^\circ$$

And substitute the values of α and β into equation (6).

$$D(\alpha) = 2(76.85525^\circ) - 8(46.95076^\circ) + 3\pi$$

$$D(\alpha) = 318.10442^\circ$$

$$\theta = \pi - 318.10442^\circ \approx -138.1^\circ$$

Due to its direction, we just have to subtract the value of θ from π.

$$\theta = \pi - 138.1^\circ = 41.8^\circ$$

We then do the same for orange:

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{15}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33322^2 - 1}{15}}\right)$$

$$\alpha \approx 76.84032^{\circ}$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(76.84032^{\circ})}{1.33322}\right)$$

$$\beta \approx 46.91713^{\circ}$$

$$D(\alpha) = 2(76.84032^{\circ}) - 8(46.91713^{\circ}) + 3\pi$$

$$D(\alpha) = 318.34360^{\circ}$$

$$\theta = \pi = 318.34360^{\circ} \approx -138.3^{\circ}$$

$$\theta = \pi - 138.3^{\circ} = 41.7^{\circ}$$

And yellow:

$$\alpha = \arccos\left(\sqrt{\frac{n^2 - 1}{15}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33472^2 - 1}{15}}\right)$$

$$\alpha \approx 76.80588^{\circ}$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(76.80588^{\circ})}{1.33472}\right)$$

$$\beta \approx 46.83973^{\circ}$$

$$D(\alpha) = 2(76.80588^{\circ}) - 8(46.83973^{\circ}) + 3\pi$$

$$D(\alpha) = 318.89392^{\circ}$$

$$\theta = \pi - 318.89392^{\circ} \approx -138.9^{\circ}$$

$$\theta = \pi - 138.9^{\circ} = 41.1^{\circ}$$

Then, just as in previous calculations, do the same for the rest of the colors. The values I had acquired are put into this table:

Colors	Angle of elevation/°		
	Primary rainbow	Secondary rainbow	Tertiary rainbow
Red	42.1	50.8	41.8
Orange	42.0	50.9	41.7
Yellow	41.8	51.3	41.1
Green	41.6	51.8	40.4
Blue	41.5	52.5	39.5
Indigo	40.7	53.3	38.3
Violet	40.4	53.9	37.6

From these values, we can see that the order of the colors for the tertiary rainbow is the same as the primary rainbow, and that it is at roughly the same height as the primary rainbow, only in the opposite direction of it.

VI. General Conjecture

From the calculations that I have done, I have made a sort of conjecture for the general formula of any order of rainbows. As can be seen from the diagrams previous, if lines were to be drawn from the points the light makes contact with the circumference of the sphere to the origin, O, then isosceles triangles will be formed. These triangles are isosceles because the feet of each triangle equals to the radius of the radius of the spherical water droplet. Therefore, the triangles formed have the angles $\beta°$, $\beta°$ and $\pi - 2\beta°$. The angles of rotation remain the same throughout the previous calculations because α and β are variables to represent the refractive angle and the incident angle respectively.

Within the assumption that the trend will continue, the general formula that can be derived from this pattern is:

$$D(\alpha) = 2(\alpha - \beta) + t(\pi - 2\beta)$$

Where t is the number of total internal reflections that the ray undergoes inside the water droplet.

Finding the Quaternary Rainbow

We shall use this conjecture to find for the quaternary rainbow. Since the primary, secondary and tertiary rainbows have seemingly been named for the number of total internal reflections that occur inside the water droplet, then the quaternary rainbow (also called the fourth-order rainbow) will have four internal reflections. Therefore, we substitute 4 into t.

$$D(\alpha) = 2(\alpha - \beta) + 4(\pi - 2\beta)$$

$$D(\alpha) = 2\alpha - 2\beta + 4\pi - 8\beta$$

$$D(\alpha) = 2\alpha - 10\beta + 4\pi \qquad (8)$$

As we had done for the previous calculations, we substitute in the value of β for equation (1) and then differentiate to find the minimum value of $D(\alpha)$. Which means finding the value of α for when $D'(\alpha) = 0$.

$$D'(\alpha) = 2 - 10\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}}$$

$$2 - 10\frac{\cos(\alpha)}{n\sqrt{1 - \frac{sin^2(\alpha)}{n^2}}} = 0$$

$$10\frac{\cos(\alpha)}{n\sqrt{1-\frac{sin^2(\alpha)}{n^2}}}=2$$

$$\cos(\alpha)=\frac{n\sqrt{1-\frac{sin^2(\alpha)}{n^2}}}{5}$$

$$\cos^2(\alpha)=\frac{n^2\left(1-\frac{sin^2(\alpha)}{n^2}\right)}{25}$$

$$25\cos^2(\alpha)=n^2-sin^2(\alpha)$$

$$n^2=25\cos^2(\alpha)+sin^2(\alpha)$$

$$n^2=24\cos^2(\alpha)+1$$

$$24\cos^2(\alpha)=n^2-1$$

$$\cos^2(\alpha)=\frac{n^2-1}{24}$$

$$\cos(\alpha)=\sqrt{\frac{n^2-1}{24}}$$

$$\alpha=\arccos\left(\sqrt{\frac{n^2-1}{24}}\right) \quad (9)$$

We then substitute in the value of the refractive index of water for *n* and find the general angle at which the rainbow lies. So we then find for α using equation (9).

$$\alpha=\arccos\left(\sqrt{\frac{1.333^2-1}{24}}\right)$$

$$\alpha\approx 79.63503°$$

And then we use this value to find for β.

$$\beta=\arcsin\left(\frac{\sin(79.63503°)}{1.333}\right)$$

$$\beta\approx 47.55672°$$

Then, to find the minimum angle of deviation, we substitute both values into equation (8).

$$D(\alpha)=2(79.63503°)-10(47.55672°)+4\pi$$

$$D(\alpha)=403.70286°$$

$$\theta = \pi - 403.70286^\circ = -223.70286^\circ = 223.70286^\circ$$

$$\theta = \pi - 223.70286^\circ \approx -43.7^\circ = 43.7^\circ$$

As with the tertiary rainbow, we do not regard the negative sign and subtract the value from π to retrieve the angle of elevation. We are then left with the value of 43.7° as the angle of elevation.

We then try to find for the angle of elevation of the seven colors of the rainbow.

Firstly, we shall find for red. Using the same methods as previous, we will first find for the values of α and β using equations (9) and (1), and then we will use both values to find the minimum angle of deviation and, finally, the angle of elevation.

$$\alpha = \arccos\left(\sqrt{\frac{n^2-1}{24}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33257^2-1}{24}}\right)$$

$$\alpha \approx 79.64277^\circ$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(79.64277^\circ)}{1.33257}\right)$$

$$\beta \approx 47.57848^\circ$$

$$D(\alpha) = 2(79.64277^\circ) - 10(47.57848^\circ) + 4\pi$$

$$D(\alpha) = 403.50074^\circ$$

After finding the $D(\alpha)$, we then calculate the angle of elevation.

$$\theta = \pi - 403.50074^\circ = -223.50074^\circ$$

$$\theta = \pi - 223.50074^\circ \approx -43.5^\circ = 43.5^\circ$$

We then calculate for orange:

$$\alpha = \arccos\left(\sqrt{\frac{n^2-1}{24}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33322^2-1}{24}}\right)$$

$$\alpha \approx 79.63108°$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(79.63108°)}{1.33322}\right)$$

$$\beta \approx 47.54559°$$

$$D(\alpha) = 2(79.63108°) - 10(47.54559°) + 4\pi$$

$$D(\alpha) = 403.80626°$$

$$\theta = \pi - 403.80626° = -223.80626°$$

$$\theta = \pi - 223.80626° \approx -43.8° = 43.8°$$

And for yellow:

$$\alpha = \arccos\left(\sqrt{\frac{n^2-1}{24}}\right)$$

$$\alpha = \arccos\left(\sqrt{\frac{1.33472^2-1}{24}}\right)$$

$$\alpha \approx 79.60413°$$

$$\beta = \arcsin\left(\frac{\sin(\alpha)}{n}\right)$$

$$\beta = \arcsin\left(\frac{\sin(79.60413°)}{1.33472}\right)$$

$$\beta \approx 47.46987°$$

$$D(\alpha) = 2(79.60413°) - 10(47.46987°) + 4\pi$$

$$D(\alpha) = 404.50956°$$

$$\theta = \pi - 404.50956° = -224.50956°$$

$$\theta = \pi - 224.50956° \approx -43.8° = 43.8°$$

And then we find the angles of elevation for blue, green, indigo and violet. The results for each of these colors are shown in the table below, along with the values for the primary, secondary and tertiary rainbows:

Colors	Angle of elevation/°			
	Primary rainbow	Secondary rainbow	Tertiary rainbow	Quaternary rainbow
Red	42.1	50.8	41.8	43.5
Orange	42.0	50.9	41.7	43.8
Yellow	41.8	51.3	41.1	44.5
Green	41.6	51.8	40.4	45.4
Blue	41.5	52.5	39.5	46.5
Indigo	40.7	53.3	38.3	48.0
Violet	40.4	53.9	37.6	49.0

Looking at the values, we can see that the quaternary rainbow mimics the secondary rainbow as it is placed higher than the tertiary rainbow and has an inverted order of colors compared to the tertiary rainbow.

However, due to the already elusive nature of the tertiary rainbow, finding the quaternary rainbow in practice is even harder. I have not found any photographic evidence for quaternary rainbows. I have found, however, the angle of elevation in which an observer can see a quaternary rainbow. Quoting, again, Harvard Physics, the angle seems to be around 44°, which roughly matches with the result that I have achieved. And thus the conjecture seems somewhat correct.

VII. Conclusion

Going back to my research question: "How does the order of the rainbow affect the position of its constituent colors as well as its position in the sky?" I have found that there is no real pattern between the angles of the colors to the order of the rainbows. Except for the colors red and orange, the distance between each angle continues getting larger as the order gets higher. From this we can see that the pattern seem to pertain that for each odd-order rainbow it goes from red to violet, and for each even-order rainbow it goes from violet to red.

A pattern can be seen in the position of the rainbows in the sky in terms of the order of the rainbows. Primary and secondary rainbows appear in front of the observer, while tertiary and quartenary rainbows appear behind the observer. Based on these observations, the rainbows lie parallel to each other, primary rainbow with tertiary rainbow and secondary rainbow and quartenary rainbow. We may make a hypothesis regarding the position and the order of the rainbows.

There may be exceptions to this observation, such as when the order increases to the point where the colors blend together, but due to the time constraints I am unable to put more research into it.

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